# ON THE SELF-DECOMPOSABILITY OF THE FRÉCHET DISTRIBUTION

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Abstract. Let $\{\Gamma_t,\, t \geq 0\}$ be the Gamma subordinator. Using a moment identification due to Bertoin-Yor (2002), we observe that for every $t > 0$ and $\alpha \in (0,1)$ the random variable $\Gamma_t^{-\alpha}$ is distributed as the exponential functional of some spectrally negative Lévy process. This entails that all size-biased samplings of Fréchet distributions are self-decomposable and that the extreme value distribution $F_\xi$ is infinitely divisible if and only if $\xi \notin (0,1)$, solving problems raised by Steutel (1973) and Bondesson (1992). We also review different analytical and probabilistic interpretations of the infinite divisibility of $\Gamma_t^{-\alpha}$ for $t, \alpha > 0$.

## 1. Introduction

The extreme value theorem - see e.g. Theorem 8.13.1 in [4] - states that non-degenerate distribution functions arising as limits of properly renormalized running maxima of i.i.d. random variables belong to one of the families

$$F_0(x) \;=\; e^{-e^{-x}},\; x \in \mathbb{R}, \qquad \text{or} \qquad F_\xi(x) \;=\; \begin{cases} 1 - e^{-x^{1/\xi}} & \text{if } \xi > 0 \\ e^{-x^{1/\xi}} & \text{if } \xi < 0 \end{cases},\; x > 0.$$

The distribution $F_0$ is known as the Gumbel distribution, whereas $F_\xi$ is called a Weibull distribution for $\xi > 0$ and a Fréchet distribution for $\xi < 0$. In the following, we denote by $X_\xi$ the random variable with distribution function $F_\xi$. Observe that

$$\frac{1 - X_\xi}{\xi} \;\xrightarrow{d}\; X_0 \qquad \text{as } \xi \to 0,$$

so that the above parametrization is continuous in $\xi$. In the present paper we are interested in the self-decomposability (SD) of $X_\xi$, referring e.g. to Section 15 in [14] for an account on self-decomposability. The Gumbel distribution is SD because of the identities

$$X_0 \;\stackrel{d}{=}\; -\log L \;\stackrel{d}{=} -\alpha \log L \,+\, \alpha \log S_\alpha$$

for every $\alpha \in (0,1)$, where here and throughout $L$ stands for the standard exponential variable and $S_\alpha$ for the standard positive $\alpha$-stable variable - see e.g. Exercise 29.16 in [14] for a proof of the second identity. If $\xi \in (0,1)$ then the variable $X_\xi$ is not infinitely divisible (ID) and hence not SD, because of its superexponential distribution tails - see e.g. Theorem 26.1 in [14]. When $\xi \geq 1$, the variable $X_\xi$ has a completely monotone density and is ID by Goldie's criterion - see e.g. Theorem 4.2 in [17], or by the ME property which makes it the

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first-passage time of some continuous time Markov chain - see e.g. Chapter 9 in [5] for an account. When $|\xi| \geq 1$, the identity in law

$$X_\xi \stackrel{d}{=} L^\xi$$

and the HCM theory of Thorin and Bondesson [5] show that the distribution of $X_\xi$ is a generalized Gamma convolution (GGC) and is hence SD - see Example 4.3.4 in [5]. The natural question whether $X_\xi$ is SD or even ID for $\xi \in (-1, 0)$ was first raised by Steutel in 1973 - see Section 3.4 in [17], and has remained open ever since. In section 4.5 of [5] - see also the Appendix B.3 of [18], this problem is rephrased in the broader context of generalized Gamma distributions. The latter are power transformations of $\Gamma_t$ where $\{\Gamma_t,\, t \geq 0\}$ is the Gamma subordinator, and can be thought of as size-biased samplings of $X_\xi$ when $\xi < 0$, in view of the formulæ

$$\mathbb{E}[f(\Gamma_t^\xi)] \;=\; \frac{\mathbb{E}[f(X_\xi)X_\xi^u]}{\mathbb{E}[X_\xi^u]}$$

valid for every $f$ bounded continuous and $t > 0$, with $u = (t-1)/\xi$. Recall in passing that Steutel's equation - see e.g. Theorem 51.1 in [14] - establishes a precise link between size-biased sampling of order one and infinite divisibility for integrable positive random variables. In this note, we provide an answer to the above questions of [17, 5].

**Theorem.** *For every $\xi \in (-1, 0)$ and $t > 0$, the random variable $\Gamma_t^\xi$ is* SD.

As a direct consequence of this result, all Fréchet distributions are SD and the extreme value distribution $F_\xi$ is ID if and only if $\xi \notin (0, 1)$. Contrary to the case $|\xi| \geq 1$, our argument is probabilistic and consists in showing that $\Gamma_t^\xi$ is distributed as the exponential functional of some spectrally negative Lévy process. This extends a classical result of Dufresne [6] for the case $\xi = -1$. The identification is made possible thanks to a entire moment method due to Bertoin-Yor [3], which applies in our context as a case study. The proof is given in the next section.

In Section 3, we review the possible interpretations of the infinite divisibility of $\Gamma_t^\xi$ for $\xi < 0$. The classical case $\xi = -1$ allows at least four different formulations in terms of processes, and also an explicit computation of the Lévy density which shows the GGC property without the HCM argument. For $\xi < -1$ the ID property is only known by analytical means and there is no direct probabilistic explanation, save for the case $t = 1$ by subordination or, tentatively, the spectral theory of a certain spectrally positive Markov processes. The situation for $\xi \in (-1, 0)$ is exactly the opposite since in addition to the exponential functional argument, the ID property can also be obtained rigorously by a first-passage time argument for a spectrally positive Markov processes. On the other hand there is no analytic proof of the ID property for $\xi \in (-1, 0)$. In this situation the GGC character of $\Gamma_t^\xi$ remains in particular an open question, which we plan to tackle in some further research.

## 2. Proof of the Theorem

We begin with a computation on the Gamma function.

**Lemma.** *For every $\alpha \in (0,1)$ and $u, t > 0$ one has*

$$\frac{u\Gamma(t+\alpha(u+1))}{\Gamma(t+\alpha u)} \;=\; \left(\frac{\Gamma(t+\alpha)}{\Gamma(t)}\right) u \;+\; \int_{-\infty}^{0} (e^{ux} - 1 - ux) f_{\alpha,t}(x) dx,$$

*where*

$$f_{\alpha,t}(x) \;=\; \frac{e^{(1+t/\alpha)x}(\alpha + e^{x/\alpha} + t(1-e^{x/\alpha}))}{\alpha\Gamma(1-\alpha)(1-e^{x/\alpha})^{\alpha+2}}$$

*is the density of a Lévy measure on $(-\infty, 0)$.*

*Proof.* We set $\lambda = t + \alpha u > 0$ and compute

$$\begin{aligned}
\frac{\Gamma(\lambda+\alpha)}{\Gamma(\lambda)} &= \frac{\lambda\beta(\lambda+\alpha, 1-\alpha)}{\Gamma(1-\alpha)} \\
&= \frac{\lambda}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{e^{-(\alpha+\lambda)x}}{(1-e^{-x})^{\alpha}}\, dx \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} (1-e^{-\lambda x}) \frac{e^{-\alpha x}}{(1-e^{-x})^{\alpha+1}}\, dx
\end{aligned}$$

where the second equality comes from a change of variable and the third from an integration by parts. This yields

$$\begin{aligned}
\frac{u\Gamma(t+\alpha(u+1))}{\Gamma(t+\alpha u)} &= \frac{\alpha u}{\Gamma(1-\alpha)} \int_0^{+\infty} (1-e^{-(t+\alpha u)x}) \frac{e^{-\alpha x}}{(1-e^{-x})^{\alpha+1}}\, dx \\
&= \left(\frac{\Gamma(t+\alpha)}{\Gamma(t)}\right) u \;+\; \frac{\alpha u}{\Gamma(1-\alpha)} \int_{-\infty}^{0} (1-e^{\alpha u x}) \frac{e^{(\alpha+t)x}}{(1-e^{x})^{\alpha+1}}\, dx \\
&= \left(\frac{\Gamma(t+\alpha)}{\Gamma(t)}\right) u \;+\; \int_{-\infty}^{0} (e^{ux} - 1 - ux) f_{\alpha,t}(x) dx
\end{aligned}$$

where again, the second equality comes from a change of variable and the third from an integration by parts.

□

**Remarks.** (a) The above proof follows [2] p. 102. Notice in passing that some computations performed in [2] are slightly erroneous. For example the subordinator whose exponential functional is distributed as $\tau_\alpha^{-\alpha}$ (with the notation of [2]) has no drift, but it is also killed at rate $1/\Gamma(1-\alpha)$.

(b) The above decomposition extends to $\alpha = 1$ since

$$\frac{u\Gamma(t+(u+1))}{\Gamma(t+u)} \;=\; u(t+u)$$

is the Lévy-Khintchine exponent of a drifted Brownian motion (the latter was already noticed in [3] - see Example 3 therein - in order to recover Dufresne's identity). However, such a formula does not seem to exist for $\alpha > 1$.

**End of the proof.** Fix $\xi \in (-1,0), t > 0$, and set $\alpha = -\xi \in (0,1)$ for simplicity. The entire moments of $\Gamma_t^\alpha$ are given for every $n \geq 1$ by

$$\begin{aligned}\mathbb{E}[\Gamma_t^{\alpha n}] &= \frac{\Gamma(t+\alpha n)}{\Gamma(t)}\\ &= \frac{\Gamma(t+\alpha)}{\Gamma(t)} \times \cdots \times \frac{\Gamma(t+\alpha n)}{\Gamma(t+\alpha(n-1))} \;=\; m\,\frac{\psi(1)\ldots\psi(n-1)}{(n-1)!}\end{aligned}$$

with the notation

$$\psi(u) \;=\; \frac{u\Gamma(t+\alpha(u+1))}{\Gamma(t+\alpha u)} \;=\; \left(\frac{\Gamma(t+\alpha)}{\Gamma(t)}\right) u \;+\; \int_{-\infty}^0 (e^{ux} - 1 - ux) f_{\alpha,t}(x)dx$$

by the Lemma, and

$$m \;=\; \frac{\Gamma(t+\alpha)}{\Gamma(t)} = \psi'(0+).$$

It is clear that $\psi$ is the Lévy-Khintchine exponent of a spectrally negative Lévy process $Z$ with infinite variation and mean $m > 0$. By Proposition 2 in [3], this entails

$$\mathbb{E}[\Gamma_t^{\alpha n}] \;=\; \mathbb{E}[I^{-n}]$$

for every $n \geq 1$, where $I$ is the exponential functional of $Z$:

$$I \;=\; \int_0^\infty e^{-Z_s}\, ds.$$

Since $Z$ has no positive jumps, Proposition 2 in [3] shows also that the random variable $1/I$ is moment-determinate, whence

$$\Gamma_t^\xi \;\stackrel{d}{=}\; I.$$

The self-decomposability of $I$ is a direct consequence of the Markov property. More precisely, introducing the stopping-time $T_y = \inf\{s > 0,\; Z_s = y\}$ for every $y > 0$, the fact that $Z_s \to +\infty$ a.s. as $s \to +\infty$ and the absence of positive jumps entail that $T_y < +\infty$ a.s. Decomposing, we get

$$I \;=\; \int_0^{T_y} e^{-Z_s}\, ds \;+\; \int_{T_y}^\infty e^{-Z_s}\, ds \;\stackrel{d}{=}\; \int_0^{T_y} e^{-Z_s}\, ds \;+\; e^{-y}\int_0^\infty e^{-Z'_s}\, ds$$

where $Z'$ is an independent copy of $Z$ and the second equality follows from the Markov property at $T_y$. This shows that for every $c \in (0,1)$ there is an independent factorization

$$I \;=\; cI \;+\; I_c$$

for some random variable $I_c$, in other words that $I \stackrel{d}{=} \Gamma_t^\xi$ is self-decomposable.

□

**Remarks.** (a) By the above Remark 1 (b), it does not seem that $\Gamma_t^\xi$ is distributed as the exponential functional of a Lévy process for $\xi < -1$. It would be interesting to have an explanation of the infinite divisibility of $\Gamma_t^\xi$ in terms of processes when $\xi < -1$. See next section for a more precise conjecture in the case $t = 1$.

(b) The self-decomposability of $S_\alpha^\alpha$ for every $\alpha \in (0,1)$ has been shown by Patie [12] in using the same kind of argument. Specifically, one can write

$$\mathbb{E}[S_\alpha^{n\alpha}] \;=\; \frac{\Gamma(1+n)}{\Gamma(1+\alpha n)} \;=\; m\,\frac{\psi(1)\ldots\psi(n-1)}{(n-1)!}$$

where we use the same notation as above and, correcting small mistakes made in Paragraph 3.2 of [12],

$$\psi(u) \;=\; \frac{u}{\Gamma(1+\alpha)} \;+\; \int_{-\infty}^0 (e^{ux}-1-ux)\frac{(1-\alpha)e^{x/\alpha}((2-\alpha)e^{x/\alpha}+(1-e^{x/\alpha}))}{\alpha^2\Gamma(1+\alpha)(1-e^{x/\alpha})^{3-\alpha}}\,dx$$

is the Lévy-Khintchine exponent of some spectrally negative Lévy process with positive mean. Setting $\alpha = t = 1/2$ and comparing the above expression to the one in the Lemma, one can check the well-known identity in law

$$\sqrt{S_{1/2}} \;=\; \frac{1}{2\sqrt{\Gamma_{1/2}}}. \tag{2.1}$$

The present paper shows that all positive powers of $S_{1/2}$ are SD and one may wonder if the same is true for $S_\alpha$ with any $\alpha \in (0,1)$. See [9] for related results and also for a characterization of the SD property of negative powers of $S_\alpha$ when $\alpha \leq 1/2$.

## 3. Further remarks and open questions

In this section we would like to review several existing or tentative approaches for the ID, SD and GGC properties of the distribution of $\Gamma_t^\xi$ or $X_\xi$ when $\xi \leq 0$.

3.1. **The case $\xi = 0$.** This is a rather specific situation but we include it here for completeness. As mentioned in Section 3.4 of [17], the SD property of the two-sided $X_0$ is a direct consequence of the extreme value theory because

$$L_1 + \frac{L_2}{2} + \cdots + \frac{L_n}{n} - \log n \;\stackrel{d}{=}\; \max(L_1,\ldots,L_n) - \log n \;\stackrel{d}{\longrightarrow}\; X_0 \tag{3.1}$$

as $n \to +\infty$, where $L_1,\ldots,L_n$ are independent copies of $L \sim$ Exp (1). The above identity and convergence in law, readily obtained in comparing Laplace transforms and distribution functions, yield after some further computations the following closed expression for the Laplace transform of $X_0$:

$$\mathbb{E}[e^{-\lambda X_0}] \;=\; \Gamma(1+\lambda) \;=\; \exp\left[-\gamma\lambda + \int_0^\infty (e^{-\lambda x}-1+\lambda x)\frac{dx}{x(e^x-1)}\right],$$

where $\gamma$ is Euler's constant. The complete monotonicity of $1/(e^x-1)$ shows then that $X_0$ is an extended GGC in the sense of Chapter 7 in [5]. See also Exercise 18.19 in [14] and Example 7.2.3 in [5] for another argument based on Pick functions, recovering (3.1).

3.2. **The case $\xi = -1$.** This is the classical situation, very well-known, but we give some details for comparison purposes. The ID property of $X_{-1}$ can first be understood by the sole fact that

$$\lim_{n\to+\infty}\left(\frac{nx}{1+nx}\right)^n \;=\; e^{-1/x}$$

because the left-hand side is the first-passage time distribution function of a certain birth and death process - see Theorem 3.1 and (3.3) in [17]. The random variable $\Gamma_t^{-1}$ is also a GIG and is hence distributed as the unilateral first-passage time of a diffusion [1], which explains its infinite divisibility by continuity and the Markov property. More precisely one has

$$\frac{1}{4\Gamma_t} \;\stackrel{d}{=}\; \inf\{u>0,\; X_u^t \,=\, 0\} \tag{3.2}$$

where $\{X_u^t,\, u \geq 0\}$ is a Bessel process of dimension $2(1-t)$ starting from one. The SD property follows as above from Dufresne's identity [6], which reads

$$\frac{2}{\Gamma_t} \;\stackrel{d}{=}\; \int_0^\infty e^{B_u - tu/2}\, du \tag{3.3}$$

where $\{B_u,\, u \geq 0\}$ is a standard linear Brownian motion. Also, Exercise 16.4 in [14] shows that $\Gamma_t^{-1}$ is the one-dimensional marginal of a certain self-similar additive process, whence its self-decomposability by Theorem 16.1 in [14]. The link between this latter interpretation and (3.2) and (3.3) has been explained in depth in [19].

It does not seem that these four interpretations can provide any explicit information on the Lévy-Khintchine exponent of $\Gamma_t^{-1}$. But in this case analytical computations can also be performed. More precisely, taking for simplicity the same normalization as in (3.2) and setting $\varphi_t(\lambda) = -\log\mathbb{E}[e^{-\lambda/4\Gamma_t}]$, formulæ (7.12.23), (7.11.25) and (7.11.26) in [7] entail

$$\varphi_t'(\lambda) \;=\; \frac{K_{t-1}(\sqrt{\lambda})}{2\sqrt{\lambda}K_t(\sqrt{\lambda})} \tag{3.4}$$

where $K_t$ is the Macdonald function. This shows $\varphi_{1/2}'(\lambda) = 1/2\sqrt{\lambda}$ viz. $\varphi_{1/2}(\lambda) = \sqrt{\lambda}$ when $t = 1/2$, and one recovers the identity (2.1). For $t = 3/2$, one obtains

$$2\varphi_{3/2}'(\lambda) \;=\; \frac{1}{1+\sqrt{\lambda}} \;=\; \mathbb{E}[e^{-\lambda(L^2\times S_{1/2})}] \;=\; \int_0^\infty \left(\frac{1}{\lambda+x}\right)\frac{\sqrt{x}\,dx}{\pi(1+x)}$$

where the first equality follows from Formula (7.2.40) in [7], and the third equality from Exercise 29.16 in [14] and (2.2.5) in [5]. This means precisely - see (3.1.1) in [5] - that the distribution of $1/4\Gamma_{3/2}$ is a GGC with zero drift and Thorin measure

$$U_{3/2}(dx) \;=\; \frac{\sqrt{x}\,dx}{2\pi(1+x)}\cdot$$

The latter property can be extended to *all* values of $t$ thanks to a result originally due to Grosswald [8] on Student's distribution. Together with (3.4), the main theorem in [8] entails

namely that the distribution of $1/4\Gamma_t$ is a GGC with zero drift and Thorin measure

$$U_t(dx) \;=\; \frac{1}{\pi^2 x (J_t^2(\sqrt{x}) + Y_t^2(\sqrt{x}))}$$

where $J_t$ and $Y_t$ are the usual Bessel functions of the first kind - see [7] p. 4.

3.3. **The case $\xi \in (-1, 0)$.** In this situation, the present paper yields an interpretation of the self-decomposability of $\Gamma_t^\xi$ by the identification

$$\Gamma_t^\xi \;\stackrel{d}{=}\; \int_0^\infty e^{-Z_u}\, du,$$

where $Z$ is a spectrally negative Lévy process. Another explanation, similar to (3.2), can then be obtained by the Lamperti transformation - see e.g. the introduction of [3] for an account and references. More precisely, setting

$$Y_u \;=\; \exp[-Z_{\tau_u}]$$

with the notation $\tau_u = \inf\{s > 0,\ \int_0^s e^{-Z_v}\, dv \;>\; u\}$, then $Y = \{Y_u,\ 0 \le u < \Gamma_t^\xi\}$ is a spectrally positive Markov process (which is also self-similar) starting from one and we have

$$\Gamma_t^\xi \;\stackrel{d}{=}\; \inf\{u > 0,\ Y_u = 0\},$$

so that the infinite divisibility of $\Gamma_t^\xi$ (but not, or at least not directly, its self-decomposability) is a consequence of the Markov property and the absence of negative jumps for $Y$. It would be interesting to see if $\Gamma_t^\xi$ could be embedded in some self-similar additive process analogous to the Brownian escape process of the case $\xi = 1$, described in Exercise 16.4 of [14].

Our main result can also be interpreted analytically in terms of generalized Bessel functions. Setting $\alpha = -\xi$ and writing down

$$\mathbb{E}[e^{-\lambda \Gamma_t^\xi}] \;=\; \frac{1}{\alpha\Gamma(t)} \int_0^\infty x^{-t\alpha - 1} e^{-\lambda x + x^{-1/\alpha}}\, dx \;=\; \frac{Z_{1/\alpha}^{t/\alpha}(\lambda)}{\alpha\Gamma(t)} \tag{3.5}$$

with the notation (1.7.42) of [11], the infinite divisibility of $\Gamma_t^\xi$ entails that the function

$$\lambda \;\mapsto\; -\left(\frac{Z_\rho^{\nu\,\prime}(\lambda)}{Z_\rho^\nu(\lambda)}\right) \tag{3.6}$$

is completely monotone for any $\rho > 1$ and $\nu > 0$. One might ask if the latter function is also a Stieltjes transform, which is equivalent to the GGC property for the distribution of $\Gamma_t^\xi$ - see Chapter 3 in [5]. Indeed, it is very natural to conjecture such a property for $\xi \in (-1, 0)$ in view of the above cases $\xi = 0$ and $\xi = -1$. Compared to classical Bessel functions, the theory of generalized Bessel functions is however rather incomplete, and proving like in [8] that the function (3.6) is a Stieltjes transform is believed to be challenging.

3.4. **The case $\xi < -1$.** In this situation the GGC property of the distribution of $\Gamma_t^\xi$ is most quickly obtained from the HCM character of the density function - see Chapter 5 and especially Example 5.5.2 in [5]. Notice that this analytical argument extends to $\xi = -1$ but not to $\xi \in (-1,0)$ since otherwise $\Gamma_t^{-\xi}$ would also have a HCM density and hence be ID, which is false. This entails that the function in (3.6) is indeed a Stieltjes transform for any $\rho \in (0,1)$ and $\nu > 0$, and it would be interesting to identify the underlying Thorin measure as in Grosswald's theorem.

A probabilistic interpretation of the self-decomposability of $\Gamma_1^\xi = L^\xi$ can also be given by Bochner's subordination. Setting $\alpha = -1/\xi \in (0,1)$, one has indeed

$$L^\xi \stackrel{d}{=} L^{-1} \times S_\alpha \stackrel{d}{=} S^\alpha_{L^{-\alpha}}.$$

where $\{S^\alpha_u,\, u \geq 0\}$ stands for the $\alpha-$stable subordinator with marginal $S_1^\alpha \stackrel{d}{=} S_\alpha$. Since $L^{-\alpha}$ is SD by our result, this means that $L^\xi$ is the marginal of some subordinator which is itself subordinated to the $\alpha-$stable one, and Proposition 4.1. in [15] shows that $L^\xi$ is SD. Besides, setting $\varphi_\alpha$ resp. $\varphi_\xi$ for the Lévy-Khintchine exponent of $L^{-\alpha}$ resp. $L^\xi$, one deduces from Theorem 30.4 in [14] the following relationship

$$\varphi_\xi(\lambda) = \varphi_\alpha(\lambda^\alpha).$$

Another, tentative, probabilistic interpretation of the self-decomposability of $L^\xi$ could be given in terms of a certain spectrally positive Markov process. Setting $\alpha = -1/\xi$ and $y_\alpha(\lambda) = \mathbb{E}[e^{-\lambda L^\xi}]$, Theorem 4.17 p. 258 in [11] and (3.5) above show that $y_\alpha$ is a solution to the fractional differential equation

$$xD_-^{\alpha+1} y_\alpha - \alpha y_\alpha = 0,$$

where $D_-^{\alpha+1}$ is a fractional Riemann-Liouville derivative - see Section 2.1 in [11]. When $\alpha = 1$ viz. $\xi = -1$ the above amounts to a Bessel equation and Feller's theory applies, making $L^{-1}$ the first-passage time of a Bessel process of index 0 - see [10]. When $\alpha \in (0,1)$ the operator $D_-^{\alpha+1}$ is the infinitesimal generator of a spectrally positive $(1+\alpha)-$stable Lévy process reflected at its minimum, which is a spectrally positive Markov process - see Section 3 in [13] and the references therein. By downward continuity, one may wonder if $L^\xi$ cannot be viewed as the first-passage time of some scale-transformation of the latter, eventhough no Feller's theory is available for fractional operators whose order lies in $(1,2)$.

The above probabilistic interpretations do not seem to hold for $t \neq 1$. On the one hand, Theorem 4.17 in [11] yields then an equation with *two* fractional derivatives of different order for the Laplace transform of $\Gamma_t^\xi$. On the other hand, keeping the notation $\alpha = -1/\xi$, Theorem 1 in [16] shows the factorization

$$\Gamma_t^\xi = \Gamma_{\alpha t}^{-1} \times S_\alpha^{(t)}$$

where $S_\alpha^{(t)}$ is the size-biased sampling of $S_\alpha$ of order $-\alpha t$, viz.

$$\mathbb{E}[f(S_\alpha^{(t)})] = \frac{\mathbb{E}[f(S_\alpha) S_\alpha^{-\alpha t}]}{\mathbb{E}[S_\alpha^{-\alpha t}]}.$$

The GGC character of $S_\alpha^{(t)}$ follows from that of $S_\alpha$ and Theorem 6.2.4. in [5], which shows by the above case $\xi = -1$ that $\Gamma_t^\xi$ is the independent product of two SD random variables. By Theorem 16.1 in [14], this entails that there exist two independent 1-self-similar additive increasing processes $Y$ and $Z$ such that

$$\Gamma_t^\xi \stackrel{d}{=} Y_{Z_1}.$$

Unfortunately, contrary to Bochner's subordination the independent composition of two additive processes is not necessarily an additive process anymore, so that the above identity does not provide a probabilistic proof of the self-decomposability of $\Gamma_t^\xi$.

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